8. ISOMETRIES IN 3 DIMENSIONS

§8.1. Central Isometries of $\mathbb{R}^3$

In 3-dimensions there are types of isometries that are different from those we considered in 2-dimensions. For example a **screw** is a rotation about an axis followed by a translation along that axis. But here we will restrict our attention to central isometries of $\mathbb{R}^3$.

A **rotary reflection** is a rotation about an axis followed by a reflection in a plane perpendicular to that axis. A special case of a rotary reflection is the **antipodal map** $A(\mathbf{v}) = -\mathbf{v}$. This is a 180° rotation followed by a reflection in a plane perpendicular to the axis.

Theorem 1: Every central isometry of $\mathbb{R}^3$ is one of the following:

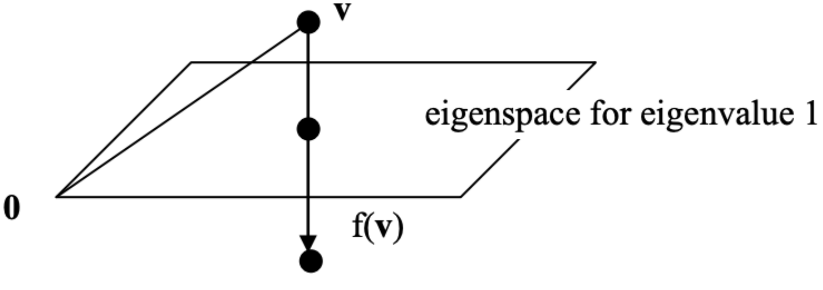
- the identity;
- a rotation;
- a reflection in a plane;
- a rotary reflection.

Proof: Let A be a central isometry of $\mathbb{R}^3$. We'll use the same symbol, A, for its matrix relative to the standard basis. Then the matrix A is orthogonal, which means that it's diagonalizable, and its eigenvalues lie on the unit circle.

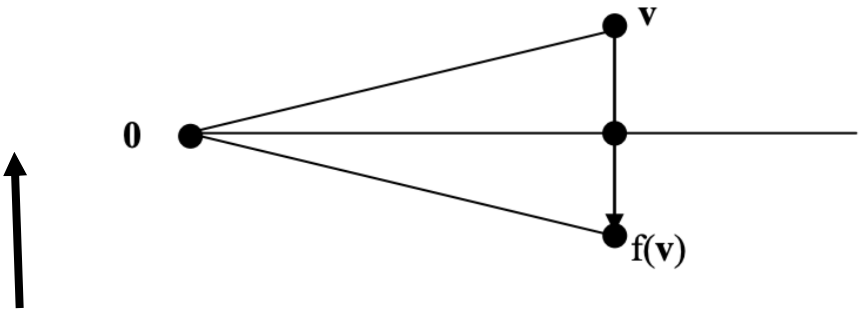
We first consider the case where the eigenvalues are real. Lying on the unit circle they must be ±1. We can choose an orthonormal basis for $\mathbb{R}^3$ that consists of three, mutually orthogonal, eigenvectors each with unit length and if we represent a typical point as (x, y, z), relative to this basis then A must map it to $(\alpha x, \beta y, \gamma z)$ where α, β and γ are the respective eigenvalues. This gives rise to the following four cases.

Case 1 (A has eigenvalues 1, 1, 1): Since A is similar to the identity matrix, A = I and so A is the identity map.

Case 2 (A has eigenvalues 1, 1, –1): If the coordinates of a point, relative to the eigenvectors, are (x, y, z) the point is mapped to $(x, y, -z)$. Here A is a reflection in the plane spanned by the eigenvectors corresponding to the eigenvalue 1.



Case 3 (A has eigenvalues 1, –1, –1): If the coordinates of a point, relative to the eigenvectors, are (x, y, z) the point is mapped to $(x, -y, -z)$. Here A is a reflection in the line spanned by an eigenvector corresponding to the eigenvalue 1. But this can also be considered as a 180° rotation about the line through that eigenvector.



Case 4 (A has eigenvalues –1, –1, –1): Here A is the antipodal map, a special case of a rotary reflection.

If the eigenvalues aren’t all real then they must consist of one real eigenvalue plus a conjugate pair of non-real ones. The real eigenvalue must be ±1 and the non-real ones must have the form $e^{\pm i\theta}$.

Case 5 (A has eigenvalues ±1, $e^{\pm i\theta}$): Relative to a corresponding (orthonormal) basis of eigenvectors

$A = \begin{pmatrix} \pm 1 & 0 \\ 0 & R \end{pmatrix}$ where R is a 2×2 real orthogonal matrix with non-real eigenvalues. Thus $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for some θ and so

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

In the first case A is a rotation about the line through the eigenvector corresponding to the eigenvalues 1. In the second case it's a rotary reflection.

§8.2. Rotations in 3 Dimensions

A **rotation** in 3 dimensions requires three parameters to specify it. We need to give the axis, which requires two vectors **a** and **b**, plus an angle θ. The two vectors need to be an ordered pair so that we can specify that the angle is given as viewed from **b** towards **a**.

A **central rotation** is one that fixes the origin. We define R(**a**, θ), where **a** is a unit vector, as the rotation about the axis that joins 0 to **a**, through the angle θ when viewed from a towards **0**. As usual anticlockwise angles are counted positive and clockwise angles are negative. Naturally we must have $\mathbf{a} \neq \mathbf{0}$ and we also we insist that $\theta \neq 0$. Rather than consider the identity isometry as a rotation through an angle of zero, about any axis, we consider it separately.

Theorem 2: The product of two rotations in $\mathbb{R}^3$ is a rotation (or the identity).

Proof: The product of two central isometries is clearly a central isometry. Moreover the determinant of this central isometry will be 1. By theorem 1 it has to be a central rotation or the identity.

The set of all central rotations, together with the identity, is a group under multiplication (following one by the other). This essentially means that the product of two rotations is a rotation, the inverse of a rotation is a rotation. This group is called the **special orthogonal group** in 3 dimensions and is denoted by **SO(3)**. It's a group that is very important in physics.

Theorem 3: Let $R(\mathbf{a}, \theta)$ be a central rotation where $\mathbf{a}$ is a unit vector and $\theta \neq 0$. Let $\mathbf{b}$ be a unit vector orthogonal to $\mathbf{a}$. Then the matrix for $R(\mathbf{a}, \theta)$ is NM^T where

$$M = (\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}) \text{ and}$$

$$N = (\mathbf{a}, \cos\theta.\mathbf{b} - \sin\theta.(\mathbf{a}\times\mathbf{b}), \sin\theta.\mathbf{b} + \cos\theta.(\mathbf{a}\times\mathbf{b})).$$

Proof: The easiest way to find the matrix of a rotation $R(\mathbf{a}, \theta)$ is to set up an orthonormal basis for $\mathbb{R}^3$ that includes the vector $\mathbf{a}$. This means that the three vectors are mutually orthogonal and all have unit length. So we choose a unit vector $\mathbf{b}$ that is orthogonal to $\mathbf{a}$. The triple $(\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b})$ will be orthonormal and moreover will be a positive triple. The matrix of $R(\mathbf{a}, \theta)$ relative to the orthonormal basis $(\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b})$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

Hence the rotation will map:

$\mathbf{a}$ to $\mathbf{a}$;

$\mathbf{b}$ to $\cos\theta.\mathbf{b} - \sin\theta.(\mathbf{a}\times\mathbf{b})$ and

$\mathbf{a}\times\mathbf{b}$ to $\sin\theta.\mathbf{b} + \cos\theta.(\mathbf{a}\times\mathbf{b})$.

Now let R be the matrix of $R(\mathbf{a}, \theta)$ relative to the original basis.

Then

$R\mathbf{a} = \mathbf{a}$;

$R\mathbf{b} = \cos\theta.\mathbf{b} - \sin\theta.(\mathbf{a} \times \mathbf{b})$;

$R(\mathbf{a} \times \mathbf{b}) = \sin\theta.\mathbf{b} + \cos\theta.(\mathbf{a} \times \mathbf{b})$.

Let $M = (\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b})$ be the matrix whose columns are the components of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a} \times \mathbf{b}$ and let

$$N = (\mathbf{a}, R\mathbf{b}, R(\mathbf{a} \times \mathbf{b})).$$

Then $RM = N$ and so $R = NM^{-1}$.

But since $\mathbf{a}$, b, $\mathbf{a} \times \mathbf{b}$ is an orthonormal basis the matrix M is orthogonal and so $M^{-1} = M^T$.

Example 1: If ρ is the rotation through 90° about (0, 1, 0) and σ is the rotation through 90° about (1, 0, 0), describe the transformation $\rho\sigma$.

Solution: $\rho\sigma$ is clearly a rotation. But we need to find its axis and the angle of rotation.

Let R and S be the respective matrices of ρ and σ relative to the standard basis of $\mathbb{R}^3$.

Then $R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$.

The matrix for $\rho\sigma$ (first ρ and then σ) will be

$SR = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Clearly $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector with corresponding eigenvalue 1.

This gives the axis of the rotation.

The vector $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

(a convenient point in the plane $x + y + z$, perpendicular to the axis) maps to $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

The angle between these is $\cos^{-1}\left(\frac{-1}{\sqrt{2}.\sqrt{2}}\right)$

$= \cos^{-1}(-1/2) = 120°$.

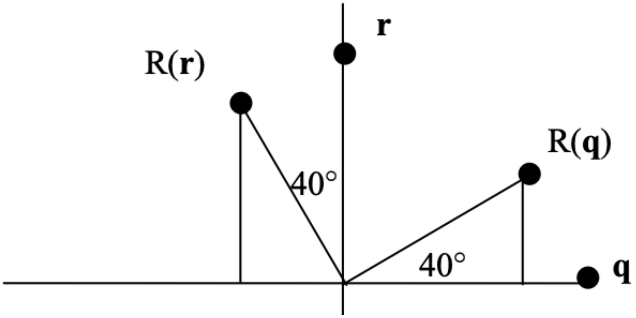
So $\rho\sigma$ is a rotation about (1, 1, 1) through ± 120°.
Which is it?

Example 2: Find the matrix of the rotation ρ about the vector (1, 1, 1) through 40°.

Solution: Let $\mathbf{p} = (1, 1, 1)$. We choose $\mathbf{q} = (1, 0, -1)$ as a convenient vector orthogonal to **p**.

Then $\mathbf{r} = \mathbf{p} \times \mathbf{q} = (-1, 2, 1)$ is mutually orthogonal to both. Moreover **p**, **q**, **r** form a positive triple by the definition of cross product.

We now make these unit vectors. For convenience we'll reuse the same names. So we now have $\mathbf{p} = \frac{1}{\sqrt{3}}(1, 1, 1)$, $\mathbf{q} = \frac{1}{\sqrt{2}}(1, 0, -1)$ and $\mathbf{r} = \frac{1}{\sqrt{6}}(-1, 2, -1)$.



Thus, if $c = \cos 40°$ and $s = \sin 40°$, $\mathrm{R}(\mathbf{p}) = \mathbf{p}$;
$\mathrm{R}(\mathbf{q}) = c\mathbf{q} + s\mathbf{r}$; $R(\mathbf{r}) = c\mathbf{r} - s\mathbf{q}$.

So $\mathbf{p} = (0.577, 0.577, 0.577)$,
$\mathbf{q} = (0.707, 0, -0.707)$,
$\mathbf{r} = (-0.408, 0.816, -0.408)$,

R(**p**) = (0.577, 0.577, 0.577),

R(**q**) = (0.279, 0.525, –0.804)

R(**r**) = (–0.767, 0.625, 0.142)

Thus:

$$R = \begin{pmatrix} 0.577 & 0.279 & -0.767 \\ 0.577 & 0.525 & 0.625 \\ 0.577 & -0.804 & 0.142 \end{pmatrix} \begin{pmatrix} 0.577 & 0.707 & -0.408 \\ 0.577 & 0 & 0.816 \\ 0.577 & -0.707 & -0.408 \end{pmatrix}^{T}$$

$$= \begin{pmatrix} 0.577 & 0.279 & -0.767 \\ 0.577 & 0.525 & 0.625 \\ 0.577 & -0.804 & 0.142 \end{pmatrix} \begin{pmatrix} 0.577 & 0.577 & 0.577 \\ 0.707 & 0 & -0.707 \\ -0.408 & 0.816 & -0.408 \end{pmatrix}$$

$$= \begin{pmatrix} 0.844 & -0.293 & 0.449 \\ 0.449 & 0.844 & -0.293 \\ -0.293 & 0.449 & 0.844 \end{pmatrix}.$$

Example 3: Find the matrix of the rotation ρ about the vector (1, 1, 1) through minus 40° (40° clockwise).

Solution: We can proceed as in example 2. However, having obtained the answer to example 3 we can observe that this rotation is simply the inverse of the one in example 2 and therefore its matrix is the inverse of the matrix obtained there.

What makes things even easier is the fact that that matrix is orthogonal and so the inverse is simply the transpose. So the required matrix is

$$\begin{pmatrix} 0.8441 & 0.4491 & -0.2931 \\ -0.2930 & 0.8441 & 0.4492 \\ 0.4491 & -0.2930 & 0.8441 \end{pmatrix}.$$

Example 4: Show that the matrix

$$R = \begin{pmatrix} 0.8441 & -0.2930 & 0.4491 \\ 0.4491 & 0.8441 & -0.2930 \\ -0.2931 & 0.4492 & 0.8441 \end{pmatrix}$$ is the matrix of a central rotation and describe this rotation geometrically.

Solution: This is the reverse to example 2.

One can check that this matrix is orthogonal and has determinant 1 and this will show that it is the matrix of a rotation about a certain axis through the origin. But since we need to find the axis, as well as the angle of rotation, we look for an eigenvector corresponding to the eigenvalue 1.

$$R - I = \begin{pmatrix} -0.1559 & -0.2930 & 0.4491 \\ 0.4491 & -0.1559 & -0.2930 \\ -0.2931 & 0.4492 & -0.1559 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1.8794 & -2.8807 \\ 0.4491 & -0.1559 & -0.2931 \\ -0.2931 & 0.4492 & -0.1559 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1.8794 & -2.8807 \\ 0 & -0.9999 & 1.0006 \\ 0 & 1.0001 & -1.0002 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1.8794 & -2.8807 \\ 0 & 1 & -1.0007 \\ 0 & 1.0001 & -1.0002 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1.8794 & -2.8807 \\ 0 & 1 & -1.0007 \\ 0 & 0 & 0.0006 \end{pmatrix}.$$

Clearly the 0.0006 should be zero and is not quite zero because of rounding errors. Similarly the −1.0007 should almost certainly be exactly –1. Making these adjustments we get $\mathrm{R} - \mathrm{I} = \begin{pmatrix} 1 & 1.8794 & -2.8807 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$.

This gives as an eigenvector $\mathbf{v} = \begin{pmatrix} 1.0013 \\ 1 \\ 1 \end{pmatrix}$ which we'll write as $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

The plane through the origin perpendicular to this is

$$x + y + z = 0.$$

We choose any non-zero vector in this plane, for example $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

Let $\mathbf{w} = \mathrm{R}\mathbf{u} = \begin{pmatrix} 1.1371 \\ -0.395 \\ -0.7423 \end{pmatrix}$.

If θ is the angle between **u** and **w** then:

$$\cos\theta = \frac{\mathbf{u}.\mathbf{w}}{|\mathbf{u}|.|\mathbf{w}|} = \frac{1.5321}{2} = 0.7661.$$

Hence θ = ± 39.9950, which is pretty close to ± 40 degrees.

Having obtained the matrix R in example 2 we know that it is a 40° anticlockwise rotation, but what if we didn't know that?

The vector $\begin{pmatrix}1\\0\\0\end{pmatrix}$ rotates to $\begin{pmatrix}0.8441\\0.4491\\-0.2931\end{pmatrix}$.

If the vectors $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0.8441\\0.4491\\-0.2931\end{pmatrix}$ form a positive triple the rotation is through plus 40 degrees. If they form a negative triple the rotation is through minus 40 degrees.

$$\begin{vmatrix}1 & 1 & 0.8441\\1 & 0 & 0.4491\\1 & 0 & -0.2931\end{vmatrix} = 0.2931 + 0.4491 = 0.7422.$$

Since this is positive the vectors form a positive triple and so the rotation is through 40 degrees clockwise when viewed from (1, 1, 1).

If we had used the matrix R in example 3 we would have had to evaluate the determinant

$$\begin{vmatrix}1 & 1 & 0.8441\\1 & 0 & -0.2930\\1 & 0 & 0.4491\end{vmatrix} = -\,0.4491 - 0.2930 = -\,0.7421.$$

The vectors here form a negative triple and so the rotation is through 40 degrees clockwise when viewed from (1, 1, 1).

The difference between the absolute values of the two determinants is merely rounding error. They should be exactly equal as they are the volumes of a parallelepiped and its mirror image.

Example 5: The matrix of the rotation through 90° about the vector (0, 3, 4) is:

$$S = \begin{pmatrix} 0 & -0.8 & 0.6 \\ 0.8 & 0.36 & 0.48 \\ -0.6 & 0.48 & 0.64 \end{pmatrix}.$$

If R is the rotation in the previous exercise describe the rotation RS by giving the axis and the angle of rotation.

Solution:

$$RS = \begin{pmatrix} 0.844 & -0.293 & 0.449 \\ 0.449 & 0.844 & -0.293 \\ -0.293 & 0.449 & 0.844 \end{pmatrix}\begin{pmatrix} 0 & -0.8 & 0.6 \\ 0.8 & 0.36 & 0.48 \\ -0.6 & 0.48 & 0.64 \end{pmatrix}$$

$$= \begin{pmatrix} -0.504 & -0.565 & 0.653 \\ 0.851 & -0.196 & 0.487 \\ -0.147 & 0.801 & 0.580 \end{pmatrix}.$$

We now find an eigenvector corresponding to the eigenvalues 1.

$$R - I = \begin{pmatrix} -1.504 & -0.565 & 0.653 \\ 0.851 & -1.196 & 0.487 \\ -0.147 & 0.801 & -0.420 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0.376 & -0.434 \\ 0.851 & -1.196 & 0.487 \\ -0.147 & 0.801 & -0.420 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0.376 & -0.434 \\ 0 & -1.516 & 0.856 \\ 0 & 0.856 & -0.484 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0.376 & -0.434 \\ 0 & 1 & -0.565 \\ 0 & 0.856 & -0.484 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0.376 & -0.434 \\ 0 & 1 & -0.565 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $z = 1$. Then $y = 0.565$, $x = 0.222$.

Thus the axis of the rotation is the line through the origin and (0.222, 0.565, 1).

The plane through the origin perpendicular to this line is $0.222x + 0.565y + z = 0$ and a suitable point on this plane is (0, 1, –0.565).

This rotates to $\begin{pmatrix} -0.504 & -0.565 & 0.653 \\ 0.851 & -0.196 & 0.487 \\ -0.147 & 0.801 & 0.580 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -0.565 \end{pmatrix}$

$$= \begin{pmatrix} -0.934 \\ -0.471 \\ 0.473 \end{pmatrix}.$$

As a check we verify that the square of the length of these two vectors is about the same: 1.32.

If θ is the angle between these vectors then:

$$\cos\theta = \frac{-0.738}{1.32} = -0.559 \text{ and hence } \theta = 124^\circ$$

(to the nearest degree).

So RS represents a rotation about the line:

$$\frac{x}{0.222} = \frac{y}{0.565} = z \text{ through the angle } 124^\circ.$$

Suppose we rotate the point **v** about the axis joining **a** to **b**, through an angle θ when viewed from **b** towards **a**. We can achieve this rotation by translating **a** to the origin, rotating about the axis that joins **0** to **b** – **a** through the angle θ, and then translating the origin back to **a**.

The resulting point will be R(**v** – **a**) + **a** where R is the matrix of R(**b** – **a**, θ). This will be the rotation through θ about some new axis.

Example 6: Rotate the point (9, 12, 15) about the axis joining (4, 5, 6) to (7, 8, 9) through the angle 40° when viewed from (7, 8, 9) towards (4, 5, 6).

Solution: After the rotation $\begin{pmatrix} 9 \\ 12 \\ 15 \end{pmatrix}$ becomes:

$$\mathbf{v} = R\left(\begin{pmatrix} 9 \\ 12 \\ 15 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}\right) + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

$= R\begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ where R is the matrix of R(**u**, 40) and

where **u** is the unit vector parallel to $\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$.

From Example 2 we see that:

$$R = \begin{pmatrix} 0.844 & -0.293 & 0.449 \\ 0.449 & 0.844 & -0.293 \\ -0.293 & 0.449 & 0.844 \end{pmatrix}.$$

Hence $\mathbf{v} = \begin{pmatrix} 6.211 \\ 5.517 \\ 9.276 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 10.211 \\ 10.517 \\ 15.276 \end{pmatrix}$.

§8.3. Platonic Solids

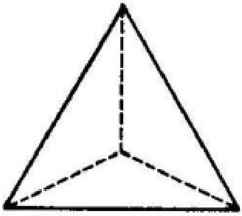
A **polyhedron** is a closed surface $\mathbb{R}^3$ made up of plane faces, each of which is a polygon. A **regular polygon** is one in which all edges have the same length and all angles are equal. We shall consider polyhedra in which the faces are regular polygons, all of the same size.

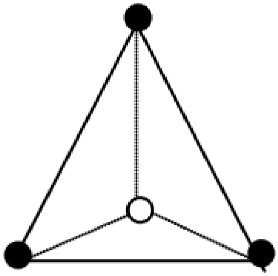
A **Platonic solid** (or regular polyhedron) is a polyhedron whose faces are identical regular polygons. The following is a list of regular polyhedra. We shall prove later that this list is complete and that only these five Platonic solids exist. Here V is the number of vertices, F is the number of faces and E is the number of edges. By Euler's Theorem:

$$V + F - E = 2.$$

Here are the 5 Platonic solids. The hidden vertices are shown as ○ .

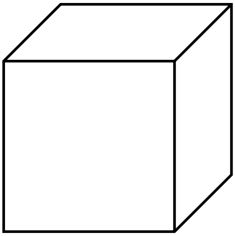
TETRAHEDRON: V = 4, E = 6, F = 4

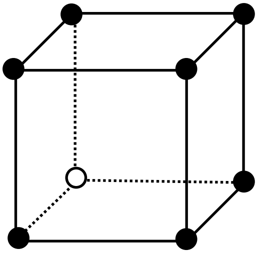




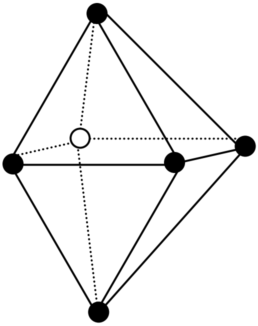
Each face is an equilateral triangle. This shape is sometimes used for cardboard drink containers.

CUBE: V = 8, E = 12, F = 6

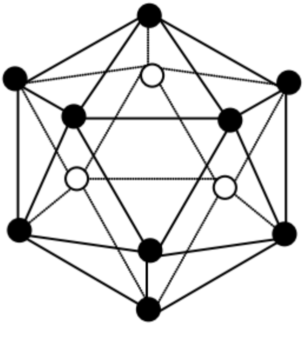


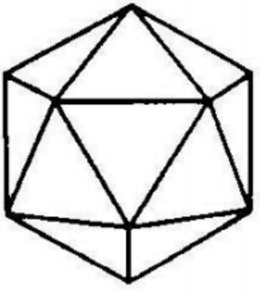


OCTAHEDRON: V = 6, E = 12, F = 8

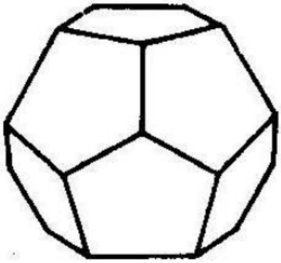


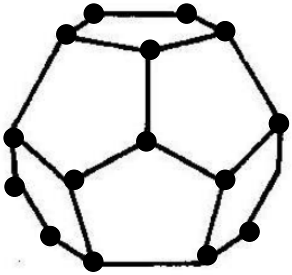
ICOSAHEDRON: V = 12, E = 30, F = 20





DODECAHEDRON: $V = 20$, $E = 30$, $F = 12$





There are 5 faces surrounding the top pentagon and 5 surrounding the bottom. These 10 faces, together with the top and bottom, make up the 12 faces. Each face has 5 edges, which would have given 60 edges, except for the fact that each edge is shared by 2 faces, so $E = 60/2$. Each face has 5 vertices, which would have given 60 vertices, except for the fact that each vertex is shared by 3 faces, so $V = 60/3 = 20$.

The dodecahedron was featured in the closing ceremony of the Sydney Olympics in 2000.

Theorem 4: There are exactly five Platonic solids: the tetrahedron, the cube, the octahedron, the icosahedron and the dodecahedron.
Proof: Let P be a polyhedron with V vertices, E edges and F faces.
Let d be the number of edges coming in to each point and let f be the number of edges around each face.

There are f edges around each face and F faces. But each edge is shared by two faces. Thus $E = \dfrac{f\,F}{2}$. Also, there are v edges around each vertex and V vertices, but each edge is shared by two vertices, so

$$E = \frac{v\,V}{2}\ .$$

Thus $f\,F = dV = 2E$.

Hence $F = \dfrac{2E}{f}$ and $V = \dfrac{2E}{d}$.

Since $V + F - E = 2$, we have $\dfrac{2E}{d} + \dfrac{2E}{f} - E = 2$.

$$\text{So } E\left(\frac{2}{d} + \frac{2}{f} - 1\right) = 2\ .$$

Now d, f are integers and each is at least 3.

Hence $\dfrac{2}{d} + \dfrac{2}{f} = 1 + \dfrac{2}{E} > 1$ and so $\dfrac{1}{d} + \dfrac{1}{f} > \dfrac{1}{2}$.

If $d = 3$, we have $\dfrac{1}{f} > \dfrac{1}{6}$ and so $f < 6$.

If $d = 4$, we have $\dfrac{1}{f} > \dfrac{1}{4}$ and so $f < 4$.

If $d = 5$, we have $\dfrac{1}{f} > \dfrac{3}{10}$ and so $f < 3\dfrac{1}{3}$.

If $d \geq 6$, we have $\dfrac{1}{f} > \dfrac{1}{3}$ and so $f < 3$, a contradiction.

So there are only 5 possibilities:

$d = 3$, $f = 3$, $E = 6$, giving $V = 4$, $F = 4$.
So P is a tetrahedron;

$d = 3$, $f = 4$, $E = 12$, giving $V = 8$, $F = 6$.
So P is a cube.

$d = 3$, $f = 5$, $E = 30$, giving $V = 20$, $F = 12$.
So P is a dodecahedron;

$d = 4$, $f = 3$, $E = 12$, giving $V = 12$, $F = 8$.
So P is an octahedron;

$d = 5$, $f = 3$, $E = 30$, giving $V = 12$, $F = 20$.
So P is an icosahedron).

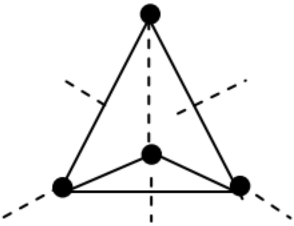
Let P be a polyhedron. The **dual** of P is the polyhedron obtained from P as follows: take a vertex in the middle of each face and join these new vertices by an edge whenever the corresponding faces are adjacent in P.

The dual of a Platonic solid is clearly a Platonic solid. The tetrahedron is self dual. The other four fall into two dual pairs. The cube and the octahedron are duals of one another, as are the dodecahedron and the icosahedron.

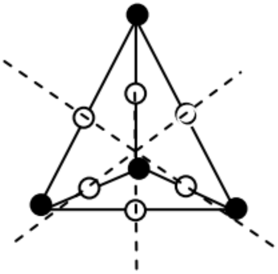
§8.4. Rotation Groups of the Platonic Solids

TETRAHEDRON:

Let T be a tetrahedron. There are four axes of 3-fold rotational symmetry, one through each vertex and the centre of the opposite face. Since two rotations share each axis, a 120° rotation and a 240° rotation, this gives eight rotations of order 3.



But, in addition, there are three axes of 2-fold symmetry (180° rotations). These join the midpoint of one edge to the midpoint of the opposite edge (the one that they don't meet). This gives an additional three rotations in the symmetry group. The identity makes the 12th element so |Rot(tetrahedron)| = 12 and |Sym(tetrahedron)| = 24.



For those who know more group theory than we've given here we point out that in fact Sym(T) is isomorphic to $\mathbf{S}_4$, the group of permutations on {1, 2, 3, 4} since the map that takes every isometry to the corresponding permutation of the four vertices is a homomorphism onto $\mathbf{S}_4$ (i.e. every permutation can be produced by an isometry) and its kernel is trivial. Since there is no mirror symmetry, Sym(T) = Rot(T) is isomorphic to $\mathbf{A}_4$ (the group of even permutations on {1, 2, 3, 4}).

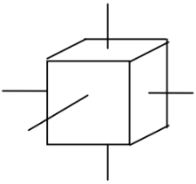
CUBE:

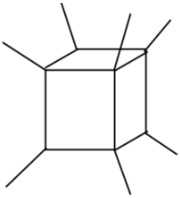
Let C be a cube. Then C has three axes of 4-fold symmetry, one through the centres of each pair of opposite faces. For each of these three axes there are three rotations, two of them 4-fold and the remaining one 2-fold. This gives altogether six 4-fold rotations and three 2-fold rotations.

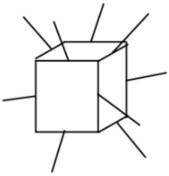
There are four axes of 3-fold rotation, one through each pair of opposite vertices, with two 3-fold rotations about each. This makes eight 3-fold rotations.

There are six axes of 2-fold rotation, one through each pair of opposite edges, giving six more 2-fold rotations. And last, but not least, is the identity making 24 rotations altogether in Rot(cube). So |Rot(C)| = 24 and |Sym(C)| = 48.

The group Rot(C) is, in fact, $\mathbf{S}_4$, the group of all permutations on {1, 2, 3, 4}. To see this we have to find four things that are permuted by the isometries in Rot(C) in all possible ways. There are six faces, but not all 6!

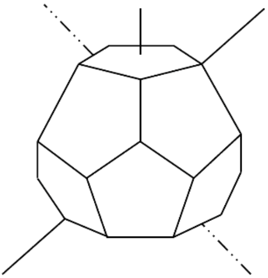






permutations are possible. For example, opposite faces must always remain opposite. There are eight vertices, but only four pairs of opposite vertices or, in other words, four diagonals. These are permuted by Rot(C) in all 4! possible ways.

DODECAHEDRON:



Let D be a dodecahedron. Then D has twelve faces, and so six axes of 5-fold rotational symmetry, one from the centre of each face to the centre of the opposite face. Each of these axes has four 5-fold rotations, giving altogether 24 rotations.

From each vertex to the opposite vertex there's an axis of 3-fold symmetry. With ten such axes and two rotations about each, we get twenty more rotations.

From the midpoint of each edge to the midpoint of the opposite edge we get an axis of 2-fold rotational symmetry. There are fifteen such axes and so fifteen more rotations.

Then, there is the identity, giving 1 + 24 + 20 + 15 = 60 elements altogether.

So |Rot(D)| = 60 and |Sym(D)| = 120.

Each of the twelve faces has five diagonals, making 60 in all. These form the vertices of five internal cubes that are permuted by the isometries in Sym(D). In this way it can be shown that Sym(D) $\cong \mathbf{S}_5$ and Rot(D) $\cong \mathbf{A}_5$.

OCTAHEDRON AND ICOSAHEDRON:
These are duals of the cube and the dodecahedron. It's not difficult to see that dual polyhedra have identical symmetry. So the symmetric group of an octahedron is S_4 and of an icosahedrons it is $\mathbf{S}_5$.

§8.5. Groups Acting on a Set

If you haven't studied much group theory, and in particular you haven't read up about G-sets, you can skip this section. If you wish to read up about G-sets you can read Chapter 10, §10.2 and §10.3 in my notes *Groups, volume 1*.

Recall that the **orbit** containing x is defined by:

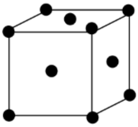
$$xG = \{xg \mid g \in G\}.$$

The set of orbits is denoted by **X/G**.

The **stabiliser** of $x \in X$ is $\boldsymbol{\sigma(x)} = \{g \in G \mid xg = x\}$.
The **fixed set** of $g \in G$ is $F(g)\ \{x \in X \mid xg = x\}$.
Remember, too, that the size of the orbit containing
$x \in X$ is the index of its stabiliser.

Also, $|X/G| = \frac{1}{|G|} \cdot \sum_{g \in G} |F(g)|$, that is, the number of orbits is the average, over G, of the numbers of elements of X fixed by the elements of G.

If a central rotation acts on the unit sphere its fixed set consists of the two points where the axis of rotation cuts the sphere. These are called the **poles** of the rotation.



Example 7: Let X be the set consisting of the corners of a cube, together with the face centres and the centre of the cube.
Clearly |X| = 15 (8 corners, 6 face centres plus the centre of the cube). The symmetry group of the cube, G, acts on X and there are 3 orbits. The eight corners form one orbit since you can get from any corner to any other by a suitable element of G. The other orbits are the six face centres and, the smallest orbit of all, consisting of just the centre of the cube.

The stabiliser of a corner is the group of order 6, consisting of the identity and the two 3-fold rotations about the diagonal through the corner as well as three reflections. The stabiliser of a face centre has order 8 (the identity, three rotations and four reflections). The stabiliser of the centre of the cube is the whole of G.

The fixed set of a reflection in the horizontal mirror plane is the set of five points on that plane. The fixed set of a 4-

fold rotation about the axis joining two face centres has three elements – those two points and the centre of the cube.

Example 8: If x is a corner of the cube, in the above example, $|xG| = 8$, since the orbit xG consists of all 8 corners. The stabilizer G_x, as we've seen, has order 6. And $|G| = 48$. Note that $8 = 48/6$.

Example 9: Suppose now that G is the rotational symmetry group of a cube, so that $|G| = 24$. It acts on the set X consisting of the centres of the 6 faces, the 8 corners, the midpoints of the 12 edges and the centre of the cube. Clearly $|X| = 27$. We can take a census of the fixed points of the elements of G.
Here n = # isometries and f = # fixed points for each.

Type	**axis**	***n***	***f***	***nf***
Identity		1	27	27
4-fold	face centres	6	3	18
2-fold	face centres	3	3	9
3-fold	corners	8	3	24
2-fold	edge midpoints	6	3	18
TOTAL		**24**		**96**

orbits = 96/24 = 4.
These orbits are:

- the centres of the faces,
- the 8 corners,

- the midpoints of the edges and
- the centre of the cube

§8.6. Finite Rotation Groups in $\mathbb{R}^3$

Theorem 7: Let G be a finite group of rotations of a subset of $\mathbb{R}^3$. Then G is either cyclic or dihedral or has order 12, 24 or 60.

Proof: We may clearly suppose that $|G| \geq 2$. Let X be the set of poles. Since G contains $|G| - 1$ rotations, each of which has two poles, $|X| \leq 2(|G| - 1)$. (Remember that some poles might be shared by more than one rotation.) Now the elements of G act on these poles. Let k be the number of orbits.

Then k is the average number of elements fixed by the elements of G.

So $k = \dfrac{2(|G| - 1) + |X|}{|G|}$ and so $|X| = (k - 2)|G| + 2$.

Now $2 \leq |X| \leq 2\,(|G| - 1)$ and hence $2 \leq k \leq 4 - \dfrac{2}{|G|}$.

Hence $k = 2$ or 3.

Case I: k = 2: Hence $|X| = 2$. Thus the non-trivial elements of G have a common axis and so, by the Lemma, G is cyclic.

Case II: k = 3: Hence $|X| = |G| + 2$. Let the three orbits be P, Q, R with sizes $p \geq q \geq r$.

Then $p + q + r = |X| = |G| + 2$.

Choose $x \in$ P, $y \in$ Q, $z \in$ R.

Their stabilizers have sizes $a = \frac{|G|}{p}$, $b = \frac{|G|}{q}$, $c = \frac{|G|}{r}$. Clearly $a \le b \le c$.

Now $\frac{|G|}{a} + \frac{|G|}{b} + \frac{|G|}{c} = p + q + r = |G| + 2$,

so $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1 + \frac{2}{|G|}$. Thus $a = 1$ or 2.

But since $|X| > 2$ the rotations in G do not all have the same axis and so the stabilizers are proper subgroups of G. Thus $b, c < |G|$ and so $a > 1$.

So $a = 2$ and hence $\frac{1}{b} + \frac{1}{c} = \frac{1}{2} + \frac{2}{|G|}$. Thus $b = 2$ or 3.

Case IIA: $a = 2, b = 2$: Hence $c = |G|/2$ and so $r = 2$. This means that $R = \{z, -z\}$ for some pole z.

The rotations in G_z have a common axis and so this subgroup is cyclic.

Let $g \notin G_z$ and suppose g has poles $\pm t$. Clearly $t \neq \pm z$ so $t \in P$ or $t \in Q$.

$\therefore |G_t| = 2$. Thus $G_t = \{I, t\}$ and so $t^2 = I$.

But also $gt \notin G_Z$ and so $(gt)^2 = I$.

This means that $gt = (gt)^{-1} = t^{-1}g^{-1} = tg^{-1}$ so $t^{-1}gt = g^{-1}$. Hence for some n, $G = \langle g, t \mid g^n = 1, t^2 = 1, t^{-1}gt = g^{-1} \rangle$ and so G is the dihedral group of order $2n$.

Case IIB: $a = 2$, $b = 3$: Thus $\frac{1}{c} = \frac{1}{6} + \frac{2}{|\mathrm{G}|}$ and so $|\mathrm{G}| = \frac{12c}{6 - c}$. Clearly $c < 6$.

Case IIB(i): $a = 2$, $b = 3$, $c = 3$: $|\mathrm{G}| = 12$.

Case IIB(ii): $a = 2$, $b = 3$, $c = 4$: $|\mathrm{G}| = 24$.

Case IIB(iii): $a = 2$, $b = 3$, $c = 5$: $|\mathrm{G}| = 60$.

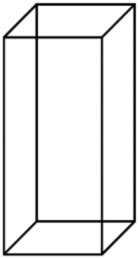
With a bit more group theory in these last three cases we could show that G is either $\mathbf{A}_4$ (the rotation group of the regular tetrahedron or octahedron), $\mathbf{S}_4$ (the rotation group of a cube) or $\mathbf{A}_5$ (the rotation group of an icosahedron or dodecahedron).

EXERCISES FOR CHAPTER 8

Exercise 1:

Identify the central isometry corresponding to the orthogonal matrix $A = \frac{1}{3}\begin{pmatrix} -2 & -2 & -1 \\ 1 & -2 & 2 \\ -2 & 1 & 2 \end{pmatrix}$

Exercise 2: Find the size of Sym(X) when X is a rectangular box of dimensions 1×1×3. Describe each of the elements of Sym(X) in terms of rotations, reflections etc, giving axes and angles of rotations and mirror planes.



Exercise 3: Find the matrix of the rotation ρ about the vector (1, 2, 3) through 25° when viewed towards the origin.

Exercise 4: The matrix of the rotation through 90° about the vector (0, 3, 4) is:

$$S = \begin{pmatrix} 0 & -0.8 & 0.6 \\ 0.8 & 0.36 & 0.48 \\ -0.6 & 0.48 & 0.64 \end{pmatrix}.$$

If R is the rotation in the previous exercise describe the rotation RS by giving the axis and the angle of rotation.

SOLUTIONS FOR CHAPTER 8

Exercise 1: We begin by finding the eigenvalues of R. They will be one third of the eigenvalues of

$$B = \begin{pmatrix} -2 & -2 & -1 \\ 1 & -2 & 2 \\ -2 & 1 & 2 \end{pmatrix}.$$

$$|\lambda I - B| = \begin{vmatrix} \lambda + 2 & 2 & 1 \\ -1 & \lambda + 2 & -2 \\ 2 & -1 & \lambda - 2 \end{vmatrix} = \lambda^3 + 2\lambda^2 - 6\lambda - 27.$$

[The easiest way to find this is to write it as: $\lambda^3 - \mathrm{tr}(B)\lambda^2 + \mathrm{tr}_2(B)\lambda - |B|$ where $\mathrm{tr}_2(B)$ is the sum of the 2×2 determinants got by deleting each row and the corresponding column. See my notes on *Matrices*.]

Factorising we get $(\lambda - 3)(\lambda^2 + 5\lambda + 9)$.

The eigenvalues of B are thus 3, $\dfrac{-5 \pm \sqrt{11}\, i}{2}$.

The eigenvalues of A are therefore 1, $\dfrac{-5 \pm \sqrt{11}\, i}{6}$.

This means that the isometry is a rotation. The axis corresponds to an eigenvector of A for the eigenvalues 1. We can take an eigenvector for B corresponding to the eigenvalues 3.

$$B - 3I = \begin{pmatrix} -5 & -2 & -1 \\ 1 & -5 & 2 \\ -2 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -5 & 2 \\ 0 & -27 & 9 \\ 0 & -9 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -5 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence the eigenvector is $\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$. The axis of the rotation is the line joining the origin to this vector.

In polar form $\dfrac{-5 \pm \sqrt{11}\,i}{6} = e^{i\theta}$ where $\cos\theta = -\frac{5}{6}$ and $\sin\theta = \dfrac{\sqrt{11}}{6}$.

In degrees, $\theta = 146.44°$. This is the angle of rotation, when viewed from (–1, 1, 3) towards the origin.

Exercise 2: There is an axis of 4-fold rotational symmetry through the centres of the square faces. This gives two 90° rotations R, R^3 plus one 180° rotation R^2. There are axes of 2-fold symmetry about the lines joining the midpoints of the longer sides to the midpoint of the opposite side. Also there is an axis of 2-fold rotational symmetry through the centres of opposite non-square faces. Together with the identity, this makes up the rotation group of order 8. Thus the order of Sym(X) is 16. The opposite isometries include reflections in the following planes:

- the vertical planes through the diagonals of the square faces;
- the vertical planes through the midpoints of opposite edges of the square faces;
- the reflection M in the horizontal plane through the centre of X.

-

This makes 5 reflections. Then there is the reflectional rotations RM and R^3M. Finally there's the antipodal isometry **v** → –**v** (which is R^2M).

Thus |Sym(X)| = **16**. The elements of Sym(X) can be summarised as follows:

TYPE	**#**
identity	1
4-fold rotations	2
2-fold rotations	5
reflections	5
rotary reflections	2
antipodal isometry	1
TOTAL	**16**

Although the question didn't ask for a presentation of Sym(X) we can provide it as:

$$\langle A, B, C \mid A^4 = B^2 = C^2 = 1, BA = A^{-1}B, AC = CA, BC = CB\rangle,$$

which is $\mathbf{D}_8 \times \mathbf{C}_2$.

Exercise 3:

Let **p** = (1, 2, 3). We choose **q** = (1, 1, –1) as a convenient vector orthogonal to **p**. Then **r** = **p** × **q** = (–5, 4, –1) is mutually orthogonal to both.

Moreover **p**, **q**, **r** form a positive triple.

We now make these unit vectors. For convenience we'll use the same names.

So we now have $\mathbf{p} = \frac{1}{\sqrt{14}}(1, 2, 3)$

$= (0.267, 0.535, 0.802)$,

$\mathbf{q} = \frac{1}{\sqrt{3}}(1, 1, -1)$

$= (0.577, 0.577, -0.577)$,

and $\mathbf{r} = \frac{1}{\sqrt{42}}(-5, 4, -1)$

$= (-0.771, 0.617, -0.154)$.

Thus, if $c = \cos 25° = 0.906$ and $s = \sin 25° = 0.423$,

$\mathrm{R}(\mathbf{p}) = \mathbf{p}$;

$\mathrm{R}(\mathbf{q}) = c\mathbf{q} + s\mathbf{r}$ and

$\mathrm{R}(\mathbf{r}) = c\mathbf{r} - s\mathbf{q}$.

So $\mathbf{p} = (0.267, 0.535, 0.802)$,

$\mathbf{q} = (0.577, 0.577, -0.577)$,

$\mathbf{r} = (-0.771, 0.617, -0.154)$,

$\mathrm{R}(\mathbf{p}) = (0.267, 0.535, 0.802)$,

$\mathrm{R}(\mathbf{q}) = (0.197, 0.784, -0.589)$

$\mathrm{R}(\mathbf{r}) = (-0.943, 0.315, 0.104)$

Thus:

$$\mathrm{R}\begin{pmatrix} 0.267 & 0.577 & -0.771 \\ 0.535 & 0.577 & 0.617 \\ 0.802 & -0.577 & -0.154 \end{pmatrix} = \begin{pmatrix} 0.267 & 0.197 & -0.943 \\ 0.535 & 0.784 & 0.315 \\ 0.802 & -0.589 & 0.104 \end{pmatrix}.$$

Hence:

$$
\begin{aligned}
\mathrm{R} &= \begin{pmatrix} 0.267 & 0.197 & -0.943 \\ 0.535 & 0.784 & 0.315 \\ 0.802 & -0.589 & 0.104 \end{pmatrix} \begin{pmatrix} 0.267 & 0.577 & -0.771 \\ 0.535 & 0.577 & 0.617 \\ 0.802 & -0.577 & -0.154 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 0.267 & 0.197 & -0.943 \\ 0.535 & 0.784 & 0.315 \\ 0.802 & -0.589 & 0.104 \end{pmatrix} \begin{pmatrix} 0.267 & 0.577 & -0.772 \\ 0.535 & 0.577 & 0.617 \\ 0.802 & -0.577 & -0.154 \end{pmatrix}^{\mathrm{T}} \\
&= \begin{pmatrix} 0.267 & 0.197 & -0.943 \\ 0.535 & 0.784 & 0.315 \\ 0.802 & -0.589 & 0.104 \end{pmatrix} \begin{pmatrix} 0.267 & 0.535 & 0.802 \\ 0.577 & 0.577 & -0.577 \\ -0.772 & 0.617 & -0.154 \end{pmatrix} \\
&= \begin{pmatrix} 0.913 & -0.325 & 0.246 \\ 0.352 & 0.933 & -0.073 \\ -0.045 & 0.024 & 0.999 \end{pmatrix}
\end{aligned}
$$

Exercise 4:

$$
\begin{aligned}
\mathrm{RS} &= \begin{pmatrix} 0.913 & -0.326 & 0.246 \\ 0.352 & 0.933 & -0.073 \\ -0.045 & 0.026 & 0.999 \end{pmatrix} \begin{pmatrix} 0 & -0.8 & 0.6 \\ 0.8 & 0.36 & 0.48 \\ -0.6 & 0.48 & 0.64 \end{pmatrix} \\
&= \begin{pmatrix} -0.408 & -0.730 & 0.549 \\ 0.790 & 0.019 & 0.612 \\ -0.579 & 0.525 & 0.625 \end{pmatrix}.
\end{aligned}
$$

We now find an eigenvector corresponding to the eigenvalues 1.

$$
\begin{pmatrix} -1.408 & -0.730 & 0.549 \\ 0.790 & -0.981 & 0.612 \\ -0.579 & 0.525 & -0.375 \end{pmatrix}
$$

$$
\to \begin{pmatrix} 1 & 0.518 & -0.390 \\ 0.790 & -0.981 & 0.612 \\ -0.579 & 0.525 & -0.375 \end{pmatrix}
$$

$$\to\begin{pmatrix} 1 & 0.518 & -0.390 \\ 0 & -0.600 & 0.920 \\ 0 & 0.825 & -0.601 \end{pmatrix}$$

$$\to\begin{pmatrix} 1 & 0.518 & -0.390 \\ 0 & -0.600 & 0.920 \\ 0 & 0.825 & -0.601 \end{pmatrix}. \text{ Let } z = 1.$$

Then $y = -0.565$, $x = 0.646$.

Thus the axis of the rotation is the line through the origin and (0.646, –0.565, 1).

The plane through the origin perpendicular to this line is z = 0.565y – 0.6460 and a suitable point on this plane is (0, 1, 0.565).

This rotates to $\begin{pmatrix} -0.504 & -0.565 & 0.653 \\ 0.851 & -0.196 & 0.487 \\ -0.147 & 0.801 & 0.580 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0.565 \end{pmatrix}$

$$= \begin{pmatrix} -0.196 \\ 0.079 \\ 1.129 \end{pmatrix}.$$

As a check we verify that the square of the length of these two vectors is the same: 1.319.

If θ is the angle between these vectors then $\cos\theta = \dfrac{0.717}{1.319}$ = 0.544 and hence θ = ± 57° (to the nearest degree).

To decide between these two we consider the vectors triple $\begin{pmatrix} 0.646 \\ -0.565 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -0.408 \\ 0.0790 \\ -0.579 \end{pmatrix}.$

Now $\begin{vmatrix} 0.646 & 1 & -0.408 \\ -0.565 & 0 & 0.0790 \\ 1 & 0 & -0.579 \end{vmatrix} = -(0.327 - 0.790) = 0.464.$

Since this is positive, the triple is a positive triple and so the rotation is through plus 57 degrees.

So RS is a rotation about the line

$\frac{x}{0.646} = -\frac{y}{0.565} = z$ through the angle 57° anticlockwise.